Quantum 1/f Effect in Resonant Biochemical Piezoelectric and MEMS Sensors

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***Abstract*—Piezoelectric sensors used for the detection of chemical agents and as electronic nose instruments are based on bulk and surface acoustic wave resonators. Adsorption of gas molecules on the surface of the polymer coating is detected by a reduction of the resonance frequency of the quartz disk, subject also to fundamental quantum 1/f frequency fluctuations.**

The quantum 1/f limit of detection is given by the quantum 1/f formula for quartz resonators. Therefore, for quantum 1/f optimization and for calculation and improvement of the fundamental sensitivity limits, we must avoid closeness of the crystal size to the phonon coherence length, which corresponds to the maximum error and minimal sensitivity situation, as shown here. Adsorbed masses below the pg range can be detected.

Microelectromechanical system (MEMS) resonators have provided a possibility for the nanominiaturization of these sensors. Essential for integrated nanotechnology, these resonant silicon bars (fingers) are excited magnetically or electrically through external applied forces, since they are not piezoelectric or magnetostrictive. The application of the quantum 1/f theory to these systems is published here for the first time. It provides simple formulas that yield much lower quantum 1/f frequency fluctuations for magnetic excitation, in comparison with electrostatically driven MEMS resonators.

I. Introduction: Conventional Quantum $1/f$ Effect, A New Aspect of Quantum Mechanics

According to the quantum 1/f effect, quantum-mechanical cross sections σ_0 and process rates Γ_0 are considered to be just the expectation values of the physical quantities σ and Γ that also contain the quantum 1/f fluctuations. Therefore,

$$S_{\delta\sigma/\sigma}(f) = S_{\delta\Gamma/\Gamma}(f) = (4\alpha/3\pi f)(\Delta\mathbf{v}/c)^2 = 4\alpha(\Delta\dot{\mathbf{P}})^2/3\pi f e^2 c^2. \quad (1)$$

Here $\alpha = (1/4\pi\varepsilon_0)(2\pi e^2/hc) = 1/137$ is Sommerfeld's fine structure constant, a dimensionless universal constant constructed from Planck's constant h, the charge e of the electron, and the speed of light c.

Phonon scattering in the resonator crystal was suspected of limiting the short- and medium-term frequency stability in all crystal resonators [1]. Phonon scattering can occur on other phonons (particularly at higher temperatures) or on crystal defects (favored by default at low temperatures). In both cases this process is shown to yield a 1/f spectrum of resonator frequency fluctuations through the conventional quantum 1/f effect. As was first shown on this basis with the help of a simple harmonic oscillator model [2], bulk acoustic wave (BAW) and surface acoustic wave (SAW) quartz resonators ought to have a Q^{-4} dependence of their FM power spectrum. This has been experimentally verified by Gagnepain and Uebersfeld for BAW resonators [3] when they first noticed their $1/Q^{4.4}$ law, and by Parker for the SAW case [4]. Although the quantum 1/f effect provided the historical basis for the derivation of the Q^{-4} law [2] as being caused by fluctuations in the dissipation rate of the quartz resonator, the exact mechanism through which the quantum 1/f effect modulates the dissipation rate remained unknown from 1978 to 1991.

Finally, the bridge directly connecting 1/f noise in frequency standards to the quantum 1/f effect in the dissipation processes was discovered [5]–[7], yielding simple quantum 1/f engineering formulas. In the present paper this thorough understanding and knowledge is used to calculate the fundamental sensitivity limits of quartz resonator sensors. If the incoherence caused by the finite phonon coherence length is considered, this can explain the experiment [8].

Here is how it works. The rate Γ of phonon-interactions which remove phonons from the main quartz sensor resonator mode is modulated by the quantum 1/f effect, therefore exhibiting observable quantum fluctuations, while its expectation value remains constant. Indeed, whenever a phonon is removed from the main resonator mode, the time-derivative of the polarization vector of the quartz crystal $d\mathbf{P}/dt = \dot{\mathbf{P}}$ is suddenly affected, suffering a step-like modification as a function of time. From Maxwell's equations we know, however, that $\dot{\mathbf{P}}$ is added to the current J (as part of the source term) and that such a sudden modification of the current causes radiation. Solving Maxwell's equations for an oscillating dipole $P(t)$ and for a sudden change $\Delta\dot{\mathbf{P}}$, we find that, as a result of the phonon removal, there is a constant energy of $(1/4\pi\varepsilon_0)(\Delta\dot{\mathbf{P}})^2/3c^3$ radiated away per unit frequency interval, i.e., per Hertz at any frequency f. Dividing this result by the energy of a photon hf, we find that there are a number of photons $2\alpha(\Delta\dot{\mathbf{P}})^2/3\pi f e^2 c^2$ emitted as bremsstrahlung of frequency f, per hertz. SI units are used here, while Gaussian units were used in [7].

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A probability proportional to $2\alpha(\Delta\dot{\mathbf{P}})^2/3\pi f e^2 c^2 \ll 1$ for the emission of a bremsstrahlung photon of frequency f exists thus per hertz, and the quartz crystal suffers a reaction, or a recoil, in its quantum state. This causes the phonon-emission rate Γ to perform quantum oscillations with frequency f and with two-sided spectral density S' of fractional rate fluctuations given by the same expression, $S'_{\delta\Gamma/\Gamma}(f) = 2\alpha(\Delta\dot{\mathbf{P}})^2/3\pi f e^2 c^2$. This is here the expression of the author's quantum 1/f noise theorem, and of the quantum 1/f effect. The one-sided spectrum is thus

$$S_{\delta\Gamma/\Gamma(f)} = 4\alpha(\Delta\dot{\mathbf{P}})^2/3\pi f e^2 c^2. \tag{2}$$

This means that any radiation caused or implied by a quantum transition from one state to another comes with a price. It reacts back on the system, causing the rate of that transition to be modulated by exhibiting observable macroscopic quantum fluctuations. These have a spectral density of fractional rate fluctuations identical to the photon-emission probability accompanying the transition considered. No knowledge of quantum mechanics is therefore needed in order to apply the quantum 1/f effect. One has to divide the energy radiated by the energy hf of one photon of frequency f. This is based just on the reality of Planck's constant h and Planck's relation between photon energy and frequency. Knowledge of electrodynamics is needed, however, in order to calculate the energy radiated in a transition.

The reader interested in a basic understanding of the quantum 1/f effect will find a most accessible description at the end of p. 8 and beginning of p. 9 in [9]. That description considers scattering of electrons as an example of transition that emits radiation and suffers a quantum 1/f modulation of its rate, rather than considering scattering of phonons, which, as we believe, is most important in crystal resonators. All that is involved in that derivation is the notion of a DeBroglie wave associated to a particle, or the notion of wave function. Old quantum mechanics notions are thus sufficient for a basic understanding of the recoil, or energy-loss mechanism, of the quantum 1/f effect. For a practical application of the quantum 1/f effect, however, classical physics is sufficient.

II. Quartz Crystal Microbalances

Resonant piezoelectric sensors are:

- used for the detection of chemical agents and for "electronic nose" instruments; based on BAW and SAW resonators.
- useful in the detection of biological agents, and can be specific, particularly if the sensitive surface is activated through deposition of a layer of the right specific antigen.

Piezoelectric sensors are of three kinds: BAW resonator-based, SAW resonator-based, and microelectromechanical system (MEMS) resonator-based (Section IV). The former are used, for instance, in the quartz crystal microbalance (QCM). This is usually a polymer-coated resonating quartz disk, a few millimeters in diameter, with smaller diameter metal electrodes on each side and with quality factor Q. Their resonance frequency is usually in the 10–30 MHz range, and 10 times higher for SAWs. SAW resonators contain interlocking comb-like fingers, between which the surface wave runs.

These sensors are based on the adsorption of many gas molecules (with total mass δm) on the surface of the polymer coating. This is detected by a reduction

$$y = \delta\omega/\omega = -\kappa\delta m/m \tag{3}$$

of the resonance frequency ω of the quartz disk. However, the latter is also subject to fundamental frequency fluctuations. Here κ is a constant of proportionality. The quantum 1/f limit of detection is given by

$$\kappa^2 S_{\delta m/m}(f) = S_y(f) = C/f;\;\; C \equiv S_y(1\text{ Hz}), \tag{4}$$

where $S_y(f)$ is given by formulas [2], [5]–[9] as

$$S_y(f) = \beta' V/fQ^4,\ \text{for } V \le \varepsilon^3;\;\; C = \beta' V/Q^4, \tag{5}$$

and

$$S_y(f) = \beta'\varepsilon^6/fVQ^4,\ \text{for } V \ge \varepsilon^3;\;\; C = \beta'\varepsilon^6/VQ^4, \tag{6}$$

as we show below. V is the quartz volume and C is a constant defined by (5) and (6). Here ε^3 is the phonon coherence volume, introduced empirically by Parker *et al.* [10], [11] as a noise coherence volume. Usually, (5) is applicable to high stability BAW resonators, whereas (6) describes SAW resonators.

The Allan (two-sample) variance of y corresponding to (4) is known to be $2C\ln 2$. It represents the flicker floor, i.e., the limit of the lowest attainable variance of the average y over an interval T. Therefore,

$$\begin{aligned}(\delta m/m)_{\min} &= (2\ln 2)^{1/2}(C)^{1/2}/\kappa \\ &= (2\ln 2)^{1/2}[S_y(1\text{ Hz})]^{1/2}/\kappa\end{aligned} \tag{4$'$}$$

is the rms fractional mass deviation attainable. This result is independent of averaging time T. However, T should be larger than the averaging time τ that reduces any coexisting white frequency fluctuation noise to this limit. The non-1/f part is covered in [1] and [12].

To optimize the device we must avoid closeness of V with ε^3, which corresponds to the maximum error and minimal sensitivity situation. (See the general graph in Fig. 1 below.)

To reduce the measurement errors, one uses differential measurements that include a reference resonator without polymer coating and subtract its frequency changes. This can effectively eliminate the destabilizing effect of temperature fluctuations, power supply instability, etc.

A. Calculation of the Quantum 1/f Sensitivity Limit

With $\langle\omega\rangle = 10^8/s$ being the average circular frequency of a thermal phonon interacting with phonons in the main

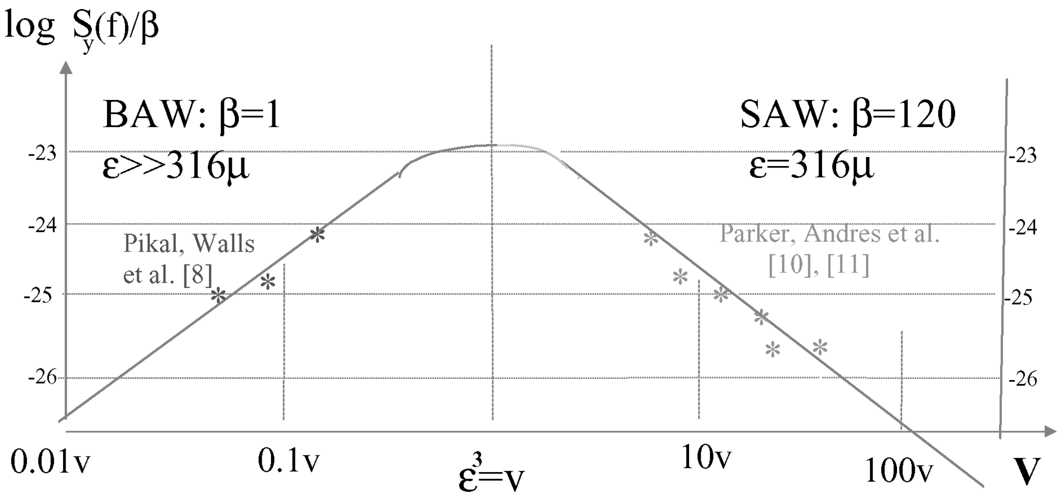

Fig. 1. Quantum 1/f frequency fluctuations of BAW and SAW resonators.

resonator mode, with $n = kT/\langle\omega\rangle\hbar$ being the average number of phonons in that typical thermal phonon mode, with $\chi = 1$ in (12), and with $T = 300K$, we obtain approximately for the quantum 1/f noise coefficient β' in (5) and (6) (see (20) below)

$$\beta' = (N/V)\alpha\hbar\langle\omega\rangle/12n\pi\chi^2mc^2 = \left(10^{22}/cm^3\right)(1/137) \times \left(10^{-27}erg\cdot s\cdot 10^8/s\right)^2/12kT\pi 10^{-27}g\cdot 9\cdot 10^{20}cm^2/s^2 = 1/cm^3. \quad (7)$$

For $V < \varepsilon^3$, this is in agreement with the known data for quartz resonators of very high Q as experimental results obtained by Pikal and Walls [8], Parker *et al.* [11] and Vig *et al.* [12], as well as other research groups, indicate. Here m is the reduced mass of the elementary oscillating dipoles, N their number in the quartz resonator volume, and χ a polarization constant of the order of the unity. This way, adsorbed masses below the $pg = 10^{-12}g$ range can be detected. We used a thermal $\langle\omega\rangle$ that radiates more than than ω in the 3-phonon process.

III. SAW Sensors

The same way, SAW resonators are used at higher frequencies in sensors that operate in the $V > \varepsilon^3$ regime. The discussion is similar, but applied to the surface and surface acoustic waves. In that case the coherence length of the phonons is smaller than the crystal size, and therefore we expect several incoherent noise contributions from various regions of the crystal.

A. Spatial Incoherence of Phonon Loss Rate Fluctuations in Low and Intermediate Q Resonators

Considering the $\nu = V/\varepsilon^3$ independently fluctuating regions similar, we replace (1) by

$$\begin{aligned} S_{\delta\Gamma/\Gamma}(f) &= \sum_{i=1}^{\nu}\langle(\delta\Gamma_i/\Gamma)^2\rangle_f = \nu^{-1}\langle(\delta\Gamma_i/\Gamma_i)^2\rangle_f \\ &= 4\alpha(\Delta\dot{\mathbf{P}}_i)^2/3\pi f\nu e^2c^2. \end{aligned} \quad (8)$$

Here we assumed that $\Gamma = \nu\Gamma_i$ and that $\langle(\delta\Gamma_i/\Gamma_i)^2\rangle_f = \nu S_{\delta\Gamma/\Gamma}(f)$ is independent of i. With $\nu = V/\varepsilon^3$ we finally obtain

$$\begin{aligned} S_{\delta\Gamma/\Gamma}(f) &= 4\alpha\varepsilon^3(\Delta\dot{\mathbf{P}}_i)^2/3\pi fVe^2c^2 \\ &= 4\alpha\varepsilon^2(\Delta\dot{\mathbf{P}}_i)^2/3\pi fAe^2c^2. \end{aligned} \quad (9)$$

B. Coherent Case of High-Q Resonators

From (2) we obtain

$$S_{\delta\Gamma/\Gamma}(f) = 4\alpha(\Delta\dot{\mathbf{P}})^2/3\pi fe^2c^2. \quad (10)$$

The vibrational energy of the crystal can be written in the form

$$W = n\hbar\langle\omega\rangle = 2(Nm/2)(dx/dt)^2 \quad (11)$$

or

$$W = (Nm/e^2)(edx/dt)^2 = (m/Ne^2)\chi^2(\dot{\mathbf{P}})^2. \quad (12)$$

The factor 2 in (11) includes the potential energy contribution. Here m is the reduced mass of the elementary oscillating dipoles, e their charge, χ a polarization constant of the order of the unity, and N their number in the resonator. Applying a variation $\Delta n = 1$ we get

$$\Delta n/n = 2|\Delta\dot{\mathbf{P}}|/|(\dot{\mathbf{P}})|, \text{ or } \Delta\dot{\mathbf{P}} = \dot{\mathbf{P}}/2n. \quad (13)$$

Solving (12) for $\dot{\mathbf{P}}$ and substituting into (13), we obtain

$$|\Delta\dot{\mathbf{P}}| = (N\hbar\langle\omega\rangle/n)^{1/2}(e/2\chi). \quad (14)$$

Substituting $\Delta\dot{\mathbf{P}}$ into (2), we get

$$\Gamma^{-2}S_\Gamma(f) = N\alpha\hbar\langle\omega\rangle/3n\pi mc^2f\chi^2 \equiv \Lambda/f. \quad (15)$$

Using the harmonic oscillator relation

$$\omega^2 = \omega_0^2 - 2\Gamma^2, \ \ \omega\delta\omega = -2\Gamma\delta\Gamma, \quad (16)$$

between resonance frequency ω and dissipation Γ, with $Q = \omega/2\Gamma$ we obtain

$$\begin{aligned} S_{\delta\omega/\omega}(f) &= (1/4Q^4)S_{\delta\gamma/\gamma}(f) \\ &= (1/4Q^4)(\Lambda/f) \\ &= N\alpha\hbar\langle\omega\rangle/12n\pi mc^2 f\chi^2 Q^4, \end{aligned} \tag{17}$$

$$S_y(f) = \beta' V/fQ^4, \text{ for } V \leq \varepsilon^3, \tag{18}$$

and

$$S_y(f) = \beta'\varepsilon^6/fVQ^4, \text{ for } V \geq \varepsilon^3, \tag{19}$$

as was stated in (5) and (6) above (Fig. 1). Here, with an intermediary value $\langle\omega\rangle = 10^8/s$, with $n = kT/\hbar\langle\omega\rangle$, $T = 300K$, and $kT = 4 \cdot 10^{-14}$ erg, we get

$$\begin{aligned} \beta' &= (N/V)\alpha\hbar\langle\omega\rangle/12n\pi\chi^2 mc^2 = (10^{22}/cm^3)(1/137) \\ &\times (10^{-27} erg \cdot s 10^8/s)^2/12kT\pi 10^{-27} g 9 \cdot 10^{20} cm^2/s^2 \\ &= 1/cm^3, \end{aligned} \tag{20}$$

as stated above in (7).

C. The Case of Defect Scattering: A Two-Phonon Process

A phonon from the main resonator mode scatters on a defect, and a phonon of comparable frequency emerges into another mode with much smaller phonon occupation number $n_\omega = kT/\hbar\omega$. In this case we have to replace $\langle\omega\rangle$ by ω and $n_{\langle\omega\rangle}$ with n_ω, which gives a β-value that is $(\langle\omega\rangle/\omega)^2$ smaller, i.e., 10^4–10^6 times smaller.

D. General Case

With $\Gamma = \Gamma' + \Gamma''$, we obtain for the combined phonon and defect scattering case, in general,

$$\beta = \beta'[\Gamma'^2 + (\omega/\langle\omega\rangle)^2\Gamma''^2]/\Gamma^2. \tag{21}$$

Although the defect scattering term is small at room temperature, it may become dominant at low temperatures, when the phonon scattering rate Γ' becomes much smaller than the defect scattering rate Γ''.

One obtains again (5) and (6) or (18) and (19), with β replacing β' in the final result, as shown in Fig. 1. Since the 1/f noise level depends on the active volume, in the coherent regime one should use the lowest overtone and smallest diameter consistent with other circuit parameters. In the incoherent (low Q) case, the opposite should be considered.

IV. Silicon MEMS Resonator Sensors

The classical forms of noise present in the frequency of MEMS resonators have been recently investigated and described in detail [12], [13]. The present paper extends this investigation to 1/f noise. That is of fundamental nature in MEMS resonators, and is given by the universal quantum 1/f effect, with some similarities to the case of quartz resonators. When the environmental fluctuations are reduced, e.g., by temperature stabilization, elimination of molecular adsorption, etc., the fundamental limitation introduced by the quantum 1/f effect becomes visible as a flicker floor in the time-domain noise representation. Being ultimately based on Heisenberg's uncertainty relations, this effect defines the quantum limit achievable at low frequencies, when the classical sources of noise have been sufficiently reduced.

The quantum 1/f effect manifests itself in MEMS resonators through conventional quantum 1/f noise. This is a quantum fluctuation present in the quantum mechanical notion of "physical cross section" σ and "physical process rate" Γ, caused by the reaction of the electromagnetic field on the charged particle that emitted it or, in general, on the system that emitted it. It is given by the spectral density of fractional fluctuations, as in (1):

$$S_{\delta\sigma/\sigma}(f) = S_{\delta\Gamma/\Gamma}(f) = (4\alpha/3\pi f)(\Delta\mathbf{v}/c)^2, \tag{22}$$

where $\Delta\mathbf{v}$ is the velocity vector change of the current carriers of charge e. As shown in the case of BAW and SAW quartz resonators, or of ferroelectrics, the rate Γ of interest for the calculation of 1/f resonance frequency fluctuations in Si MEMS is the rate of phonon removal from the main mechanical oscillation mode of the oscillating bar, finger, or cantilever. As in the above-mentioned cases, our mechanism of quantum 1/f noise generation is based on bremsstrahlung from electric or magnetic dipoles, rather than from scattering of individual carriers.

The MEMS resonators are shaped as a bar of micro- or nanometric dimensions encased (fixed) at both ends, and subject to bending strain. They are driven by an AC current I flowing in a very thin straight wire attached along the bar, in the presence of a strong constant magnetic field **B** perpendicular to it. Alternatively, they could be driven electrostatically, or capacitively, by using the wire, or a metallic thin sheet deposited along that side of the bar, as one of the two "plates" of a capacitor. An AC voltage would then be applied to the capacitor at the resonance frequency.

The approach developed for the case of resonators based on quartz and other piezoelectric substances cannot be applied to Si MEMS resonators because silicon is not piezoelectric. It could only be applied to piezoelectric MEMS resonators, using, for instance, quartz. We therefore develop a new approach in this paper, applicable to the new driven MEMS resonators. We consider below both the case of magnetic excitation and the capacitive case.

A. Magnetic Excitation of a Resonant Silicon Bar

The elementary act of dissipation is the absorption of one phonon from the main resonator mode. This can happen either through three photon processes, dominant at higher temperatures, or through two processes, dominant at low temperatures, in the presence of crystal defects of various types. In both cases this elementary act has a spontaneous "bremsstrahlung photon emission (probability) amplitude" associated with it, because it causes a sudden reduction $\Delta\dot{\mathbf{m}}$ of the rate $d\mathbf{m}/dt \equiv \dot{\mathbf{m}}$ of change of the magnetic dipole moment vector $\mathbf{m}$ of the system. The system is defined here to include the current carrying circuit, the applied magnetic field $\mathbf{B}$ with the agency generating it, plus the oscillating bar, i.e., the MEMS. The rate $\dot{\mathbf{m}}$ of change of the magnetic dipole moment $\mathbf{m}$ of the system is caused by bending oscillations of the bar with the current-carrying wire attached to its side, thus cutting the lines of force of $\mathbf{B}$ while it moves. The vector $\mathbf{m} = \mathbf{n}SI/c$ is defined by the current I enclosing the area S that has a normal of unit vector $\mathbf{n}$.

The conventional quantum 1/f noise in the rate of dissipation Γ is obtained in the centimeter-gram-second Gaussian system by simply replacing the current change $e\Delta\mathbf{v}$ with $\Delta\dot{\mathbf{m}}$,

$$S_{\delta\Gamma/\Gamma}(f) = (4\alpha/3\pi f)(\Delta\dot{\mathbf{m}}/ec)^2. \tag{23}$$

To calculate the change $\Delta\dot{\mathbf{m}}$, we note that $(\dot{\mathbf{m}})^2$ is proportional to the squared transversal vibration speed of the bar and to the total mechanical energy $W = (n + 1/2)\hbar\omega$ in the main oscillation mode of the resonator,

$$(n + 1/2)\hbar\omega = \kappa(\dot{\mathbf{m}})^2/2. \tag{24}$$

Here κ is a coefficient of proportionality easy to calculate. When a phonon is removed from the main MEMS resonator mode, the energy in (24) is reduced by the amount

$$\hbar\omega = \kappa\dot{\mathbf{m}}\Delta\dot{\mathbf{m}}. \tag{25}$$

This yields

$$(\Delta\dot{\mathbf{m}})^2 = (\hbar\omega/\kappa\dot{\mathbf{m}})^2 = \hbar\omega/(2n + 1)\kappa. \tag{26}$$

Eq. (23) becomes now

$$\begin{aligned} S_{\delta\Gamma/\Gamma}(f) &= (4\alpha/3\pi f)\hbar\omega/(2n + 1)\kappa(ec)^2 \\ &= 4\omega/3\pi f(2n + 1)\kappa c^3 \\ &= 2\hbar\omega^2/3\pi f W\kappa c^3. \end{aligned} \tag{27}$$

This quantum 1/f noise, proportional to the resonance frequency and inversely proportional to the number of quanta in the main oscillation mode, decreases when the energy in the main resonator mode increases. This is characteristic of quantum mechanical emission noise in general. With $\langle\omega\rangle = 10^{12}$rad/sec and $W = 10^{-13}$ erg we obtain about $10^{-22}/f\kappa$. The MEMS data used are arbitrary.

To estimate the coefficient κ, we write the mechanical energy, including both the equally contributing kinetic and elastic components, in the form

$$W = M_{eff}v^2/2, \tag{28}$$

where M_{eff} is a mass close to the mass M of the bar. Noting that the rate of change in the magnetic moment of the system, caused by the motion of the bar of length L with speed v, is, with constant flux,

$$\dot{\mathbf{m}} = \mathbf{n}(I/c)\frac{dS}{dt} = \mathbf{n}ILv/c \tag{29}$$

we find

$$\kappa = M_{eff}v^2/\dot{\mathbf{m}}^2 \tag{30}$$

Using (29), (30) becomes

$$\kappa = M_{eff}[c/IL]^2. \tag{31}$$

With $M_{eff} = 10^{-11}$ g, $L = 10^{-3}$ cm, and $I = 1$ mA, we obtain $\kappa = 10^3$ s^{2}cm^{-3}. Substituting into (27), we obtain

$$\begin{aligned} S_{\delta\Gamma/\Gamma}(f) &= (4\alpha/3\pi f)(IL)^2\hbar\omega/(2n + 1)M_{eff}e^2c^4 \\ &= \underline{4(IL)^2\omega/3\pi f(2n + 1)}M_{eff}\underline{c}^5 \\ &= 2(IL)^2\hbar\omega^2/3\pi f W M_{eff}c^5. \end{aligned} \tag{32}$$

For instance, using the above numerical example, with $\omega = 10^{12}s^{-1}$, and with $W = 10^{-13}$ erg, we obtain $10^{-25}/f$.

B. Electrostatic Excitation of a Resonant Silicon Bar

The elementary act of dissipation is again the absorption of one phonon from the main resonator mode. This time, the quantum 1/f noise is caused by the sudden change in the rate of change of the dipole moment $\dot{\mathbf{p}}$ of the capacitor, when the phonon is absorbed. Replacing the current change $e\Delta\mathbf{v}$ this time with $\Delta\dot{\mathbf{p}}$, we obtain

$$S_{\delta\Gamma/\Gamma}(f) = 4\alpha(\Delta\dot{\mathbf{p}})^2/3\pi f e^2c^2. \tag{33}$$

To calculate the change $\Delta\dot{\mathbf{p}}$, we note that $(\dot{\mathbf{p}})^2$ is proportional to the squared transversal vibration speed of the bar and to the total mechanical energy $W = (n + 1/2)\hbar\omega$ in the main oscillation mode of the resonator,

$$(n + 1/2)\hbar\omega = \kappa'(\dot{\mathbf{p}})^2/2. \tag{34}$$

Here κ' is a coefficient of proportionality easy to calculate. When a phonon is removed from the main MEMS resonator mode, the energy in (34) is reduced by the amount

$$\hbar\omega = \kappa'\dot{\mathbf{p}}\,\Delta\dot{\mathbf{p}}. \tag{35}$$

This yields

$$(\Delta\dot{\mathbf{p}})^2 = (\hbar\omega/\kappa'\dot{\mathbf{p}})^2 = \hbar\omega/(2n + 1)\kappa'. \tag{36}$$

Eq. (32) becomes now

$$\begin{aligned} S_{\delta\Gamma/\Gamma}(f) &= (4\alpha/3\pi f)\hbar\omega/(2n + 1)\kappa'(ec)^2 \\ &= \underline{4\omega/3\pi f(2n + 1)\kappa'}\underline{c}^3 \\ &= 2\hbar\omega^2/3\pi f W\kappa'c^3. \end{aligned} \tag{37}$$

This quantum 1/f noise, proportional to the resonance frequency and inversely proportional to the number of quanta in the main oscillation mode, decreases when the energy in the main resonator mode increases. This is characteristic of quantum mechanical emission noise in general. With $\langle\omega\rangle = 10^{12}$ rad/sec and $W = 10^{-13}$ erg we obtain about $10^{-22}/f\kappa'$.

To estimate the coefficient κ', we again write the mechanical energy in the form

$$W = M_{eff}v^2/2. \tag{38}$$

Noting that the rate of change in the electric dipole moment of the system, caused by the motion of the bar of linear charge density q, is

$$\dot{\mathbf{p}} = qLv, \tag{39}$$

we find

$$\kappa' = M_{eff}v^2/\dot{\mathbf{p}}^2 = M_{eff}/q^2L^2. \tag{40}$$

With $M_{eff} = 10^{-11}$ g, $L = 10^{-3}$ cm, and $q = 0.6$ esu/cm, we obtain $\kappa' = 2.7 \cdot 10^{-5}s^2$ cm^{-3}. Substituting into (37), we obtain

$$\begin{aligned} S_{\delta\Gamma/\Gamma}(f) &= (4\alpha/3\pi f)q^2L^2\hbar\omega/(2n+1)M_{eff}(ec)^2 \\ &= \underline{4}\underline{q}^2\underline{L}^2\underline{\omega/3\pi f(2n+1)}M_{eff}\underline{c}^3 \\ &= 2q^2L^2\hbar\omega^2/3\pi fWM_{eff}c^3. \end{aligned} \tag{41}$$

For instance, with the above numerical example, with $\langle\omega\rangle = 10^{12}s^{-1}$, and with $W = 10^{-13}$ erg, we obtain $3 \cdot 10^{-18}/f$.

Finally, the quantum 1/f frequency fluctuations can be obtained from the formula

$$S_{\delta\omega/\omega} = (1/4Q^4)S_{\delta\Gamma/\Gamma} \tag{42}$$

which is derived in the Appendix below. This yields, with (41),

$$S_{\delta\omega/\omega} = (1/2Q^4)q^2L^2\hbar\omega^2/3\pi fWM_{eff}c^3 \tag{43}$$

for the electrostatic capacitively coupled case, and, from (32),

$$S_{\delta\omega/\omega} = (1/2Q^4)(IL)^2\hbar\omega^2/3\pi fWM_{eff}c^5 \tag{44}$$

for the fractional frequency fluctuation spectrum exhibited by the magnetically driven MEMS resonator. The numerical estimations show that the electrostatically driven resonator is much noisier than the magnetically driven one, when it comes to 1/f noise. This is because the magnetic dipole radiation is much weaker, being of higher order than the electric dipole radiation.

V. Conclusions

1. The quantum 1/f limit of BAW and SAW QCMs is given by simple formulas allowing calculation of the ultimate mass and concentration threshold of detection.
2. The devices are optimized by avoiding the $V = \varepsilon^3$ region.
3. The magnetic excitation sensors based on MEMS resonators have much lower 1/f noise than electrostatically driven MEMS resonators, for the same Q. Simple formulas describe the quantum 1/f noise in both cases.

Appendix A
Frequency and Phase Fluctuations from 1/f Noise in Dissipation

Resonant systems can be described as a harmonic oscillator with losses

$$dx/dt + \gamma dx/dt + \omega_0^2 x = F(t). \tag{45}$$

The quantum 1/f fluctuations are present in the loss coefficient γ. They are given by an expression of the form

$$S_{\delta\gamma/\gamma}(f) = \Lambda/f, \tag{46}$$

where Λ is a quantum 1/f coefficient characterizing the elementary loss process.

The resonance frequency is given by

$$\omega_r^2 = \omega_0^2 + \gamma^2. \tag{47}$$

The quantum 1/f fluctuations in the resonance frequency are given by $\omega_r\delta\omega_r = -2\gamma\delta\gamma$, or

$$\delta\omega_r/\omega_r = -2(\gamma/\omega_r)^2\delta\gamma/\gamma = -(1/2Q^2)(\delta\gamma/\gamma), \tag{48}$$

where $Q = \omega_r/2\gamma$ is the quality factor. Squaring, averaging, and particularizing (25) for the unit frequency interval, we obtain the Q^{-4} law

$$S_y(f) = \langle(\delta\omega_r/\omega_r)^2\rangle_f = (1/4Q^4)\langle(\delta\gamma/\gamma)^2\rangle_f = \Lambda/4fQ^4, \tag{49}$$

where y is the fractional frequency fluctuation $\delta\omega_r/\omega_r$. This law is applicable to quartz resonators, where it was first introduced [2] in 1978. The phase noise is obtained from

$$S_\phi(f) = (\omega_r/2\pi f)^2S_y(f), \text{ or } L(f) = (1/2)S_\phi(f), \tag{50}$$

in terms of the two-sided spectral density.

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