Characteristic functional of physical thermal noise, including equilibrium quantum 1/*f* noise

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I. Introduction

In a previous paper [1], the characteristic functional of quantum $1/f$ noise was derived. The result is applicable to quantum $1/f$ noise in any cross section or process rate, and in currents and voltages. In another paper [2], the application of quantum $1/f$ noise to thermal equilibrium noise was carried out, and the quantum $1/f$ Nyquist theorem was developed. Deviations from the Gaussian form of thermal noise were expressed in terms of the characteristic function [2,3].

This present paper performs the final step in the study of thermal noise with infrared radiative corrections by calculating the characteristic functional of physical thermal noise. We call it "physical" because we refer to the actual observed thermal equilibrium (Johnson) noise, rather than to the strictly Gaussian noise expected from the Nyquist formula without infrared radiative corrections. The small quantum $1/f$ contributions come in as time-dependent infrared corrections as required by the interactions of the current carriers with the electromagnetic field, or by the reaction of the bremsstrahlung back on the current which has produced it. Knowing the characteristic functional of a process represents the highest level and most comprehensive description possible for a random process.

II. Amplitude distribution and characteristic function

In terms of current fluctuations in equilibrium, to derive the characteristic function of the physical thermal noise current variable ξ, we express it in terms of the unmodulated theoretical Nyquist noise current variable x as

$$\xi = x(G/G_o)^{1/2} = x + xy \quad , \tag{1}$$

where $y = \delta G / 2G_o$ is the fractional quantum $1/f$ fluctuation in the conductivity $G = G_o + \delta G$ of the conductor whose thermal equilibrium current fluctuations we are considering. Let the quantum $1/f$ noise variable y obey a Gaussian amplitude distribution of dispersion σ_2, and x one of dispersion σ_1. We prove now that in terms of the zeroth-order Bessel function of imaginary argument, the product $z = xy$ will have an amplitude distribution $P(z) = (1/\pi\sigma_1\sigma_2)K_o(z/\sigma_1\sigma_2)$ which has an elementary characteristic function

$$X(v) = (1 + \sigma_1^2\sigma_2^2 v^2)^{-1/2} \quad . \tag{2}$$

In order to derive the amplitude distribution of $z = xy$, we introduce $\eta = \ln z$, $\alpha = \ln x$ and $\beta = \ln y$. From $\eta = \alpha + \beta$, we obtain the distribution

$$\begin{aligned} P''(\eta)d\eta &= d\eta \int P'_1(\alpha)P'_2(\eta-\alpha)d\alpha \\ &= (2d\eta/2\pi\sigma_1\sigma_2)\int \exp[-e^{2\alpha})/2\sigma_1^2 + \alpha]\exp[-e^{2(\eta-\alpha)}/2\sigma_2^2 + (\eta-\alpha)]d\alpha \\ &= (e^{\eta}d\eta/2\pi\sigma_1\sigma_2)\int \exp[-(e^{\eta}/\sigma_1\sigma_2)\cosh\varepsilon]d\varepsilon \\ &= (e^{\eta}/\pi\sigma_1\sigma_2)K_o(e^{\eta}/\pi\sigma_1\sigma_2) \ . \end{aligned} \tag{3}$$

Here $P'_1(\alpha)$ and $P'_2(\beta)$ are derived from Gaussians of dispersion σ_1 and σ_2, respectively, $\varepsilon = 2\alpha - \eta + \ln(\sigma_1/\sigma_2)$, and K is the modified zeroth-order Bessel function. Returning to the real variable z, we obtain from $P''(\eta)d\eta = P(z)dz$

$$P(z) = (1/\pi\sigma_1\sigma_2)K_o(z/\sigma_1\sigma_2) \quad . \tag{4}$$

A Fourier transform shows that the corresponding characteristic function is elementary:

$$Z(v) = (1+\sigma_1^2\sigma_2^2 v^2)^{-1/2} \quad . \tag{5}$$

This is just what we wanted to prove.

Our next step is the calculation of the characteristic function of ξ. Due to the mutual dependence of x and z, the characteristic function of ξ differs from the product of the characteristic functions of x and z. We obtain instead the amplitude probability density function (pdf)

$$\begin{aligned} I(\xi) &= \int_{-\infty} P_1(x)P_2(\xi/x-1)(1/|x| \\ &= [\exp(-1/2\sigma_2^2)/2\pi\sigma_1\sigma_2]\int_{-\infty}\exp[-x^2/2\sigma_1^2-\xi^2/2x^2\sigma_2^2+\xi/x\sigma_2^2]dx/|x| \\ &= [\exp(-1/2\sigma_2^2)/2\pi\sigma_1\sigma_2]\int_0 \exp[-x^2/2\sigma_1^2-\xi^2/2x^2\sigma_2^2]\cosh(\xi/x\sigma_2^2)dx/x \ , \end{aligned} \tag{6}$$

which we have written down directly, but which can also be obtained by substituting $(e^{\alpha}-1)^2$ for $e^{2\alpha}$ in Eq. (3). This distribution tends to a Gaussian in the limit $\sigma_2 \to 0$, but differs slightly from a Gaussian in general. The skewness is zero, but the kurtosis exhibits a small deviation from the Gaussian amplitude pdf.

By performing a Fourier transform of Eq. (6), we find again that the characteristic function of the total physical thermal noise is elementary,

$$X(k) = (1+k^2\sigma_1^2\sigma_2^2)^{-1/2}\exp[-k^2\sigma_1^2/2(1+k^2\sigma_1^2\sigma_2^2)] \quad , \tag{7}$$

while the amplitude distribution itself can not be expressed in terms of elementary functions. Here $\sigma_1^2 = 4kTGB$ and $\sigma_2^2 = (\alpha A/2N)\int_f^F (\varepsilon/\varepsilon_o)^{\alpha A} d\varepsilon/\varepsilon$, with $B = F - f$ representing the bandwidth under consideration.

In the remaining portion of the paper, we derive the characteristic functional of thermal noise, including thermal equilibrium quantum $1/f$ noise in the conductance G. We first familiarize the reader with characteristic functionals in general.

III. Characteristic functionals

The complete characterization of a random process $j(t)$ is given by its characteristic functional [4]

$$H[x(t)] = \langle e^{i(j,x)} \rangle \quad . \tag{8}$$

In our case, $j(t) = J(t) - \langle J \rangle$ is the current (or voltage or fractional conductance) fluctuation; $x(t)$ is the argument function of the functional; and the scalar product in Hilbert space is defined as

$$(j,x) = \int_0^T j^*(t)x(t)dt \quad . \tag{9}$$

Here T is the interval of observation of the fluctuations, i.e., the length of the sample considered from the stationary random process at hand. Particularly simple, the characteristic functional of a Gaussian process is [4]

$$H_G[x(t)] = \exp[-(1/2) \langle (j,x)^2 \rangle] \quad . \tag{10}$$

For thermal short-circuit current fluctuations in a resistor of conductance G described by the simple Nyquist formula with $\sigma_1^2 = 4kTGB$, the characteristic functional is

$$H[x] = \exp[-2G\int_f^F k_B T\left|x_f(t)\right|^2 df] = \exp[-2k_B TGB\int_0^T \left|x(t)\right|^2 dt] \quad , \tag{11}$$

where x_f is the Fourier-transformed argument function. Here the two-sided frequency integral is limited to the allowed bandwidth B, and the time integral to the interval T of observation. Since $B \to \infty$, $k_B T$ must be replaced by $hf/[e^{hf/kT} - 1]$ and included in the frequency integral, while the time integral must also include the Fourier-transformed Planck kernel.

The characteristic functional of quantum $1/f$ noise affecting the conductance G has been shown to be [1]

$$H[x(\varepsilon)] = \exp[-\alpha A(G^2/N)\int(\varepsilon/\varepsilon_o)^{\alpha A}\left|x(\varepsilon)\right|^2 d\varepsilon/\varepsilon] \quad . \tag{12}$$

Here $\alpha A = (2\alpha/3\pi) < (\Delta v/c)^2 >$ is the infrared exponent, which is known to enter the expression of both the basic quantum $1/f$ noise formula and the characteristic functional of quantum $1/f$ noise in two ways: as a coefficient and as an additional exponent of the frequency f. Sommerfeld's fine structure constant $\alpha = e^2/\hbar c = 1/137$ is well known, and is Boltzmann's constant k_B. Equation (12) gives the characteristic functional of $1/f$ noise for a sample containing N current carriers with an average velocity change $< (\Delta v)^2 >$ in the scattering processes which determine their mobility. Since we are interested in the variable $y = \delta G/G_o$, we must replace G by 1/4 in Eq. (12).

For each value of the frequency, we now use Eqs. (11) and (12), repeating the steps which led to the derivation of Eq. (5). This yields the characteristic functional of the term $z = xy$ in Eq. (1),

$$\begin{aligned} Z[v(f)] &= \exp[-(1/2)\int \ln(1+\sigma_1^2\sigma_2^2 v^2)df\,] \\ &= \exp\{-(1.2)\int \ln[1+2K_B TG(\alpha A/fN)(f/f_o)^{\alpha A} v^2(f)]df\}\ . \end{aligned} \tag{13}$$

According to Eq. (1), we now have to add the Nyquist term x. Repeating for every frequency f the steps which led us to Eq. (7), we obtain now the characteristic functional of physical thermal noise ζ in the form

$$\begin{aligned} F[k(f)] &= \exp\{-(1/2)\int \ln[1+2k_B TG(\alpha A/fN)(f/f_o)^{\alpha A} k^2(f)]df\} \\ &\times \exp\{-2k_B TG\int k^2(f)[1+2k_B TG(\alpha A/fN)(f/f_o)^{\alpha A} k^2(f)]^{-1} df\}\ . \end{aligned} \tag{14}$$

This equation gives the characteristic functional of physical thermal noise for a sample containing N current carriers with an average velocity change $<(\Delta v)^2>$ in the scattering processes which determine their mobility, resulting in a well-defined infrared exponent αA.

References

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